Tutorial 1: Preferences, Utilities, and Decisions

Exercise 1: Lexicographic preferences Recall the example of lexicographic preferences from the lecture. The attributes are colour, engine type, and nationality, with the ranking

$$\text{colour} > \text{engine} > \text{nationality}$$

The ordering for each of attribute is:

- *colour*: red $\succ$ blue $\succ$ green
- *engine type*: electric $\succ$ petrol $\succ$ diesel
- *nationality*: German $\succ$ French $\succ$ UK

Define a utility function that takes three inputs (colour, engine type, and nationality) and gives as output a real number, such that this utility function corresponds to the preference ordering defined above. *Don't* do this by enumerating all 27 cases! Argue for the correctness of your function.

Solution: Define

$$\begin{aligned} U(\text{red}) = U(\text{electric}) = u(\text{German}) = 2 \\ U(\text{blue}) = U(\text{petrol}) = u(\text{French}) = 1 \\ U(\text{green}) = U(\text{diesel}) = u(\text{UK}) = 0 \end{aligned}$$

Then, define

$$u(xyz) = 100 * U(x) + 10 * U(y) + U(z),$$

where $x \in \{\text{red, blue, green}\}$, $y \in \{\text{electric, petrol, diesel}\}$, and $z \in \{\text{German, French, UK}\}$.

Observe that for all $x, x' \in \{\text{red, blue, green}\}$ with $x \neq x'$, and $y, y' \in \{\text{electric, petrol, diesel}\}$ with $y \neq y'$, and $z \in \{\text{German, French, UK}\}$ we have that

(i) $|U(y) - U(y')| > U(z)$, and

(ii) $|U(x) - U(x')| > U(y) + U(z)$.

Intuitively, this ensures that the difference in utility for higher ranked attributes can not be offset by higher utility values for the lower ranked attributes. This then yields a utility representation of lexicographic ordering specified. (A formal argument is rather tedious.)

Exercise 2: Expected Utility You have three entertainment choices: football, pub, or rowing. The utility you obtain from these will depend on whether it rains or not. The utilities you get from these outcomes are as follows:

Activity	Utility if rain	Utility if no rain
football	1	2
pub	3	0
rowing	0	1

You have a decision task ahead: to choose which activity to undertake. In what follows, let p_{rain} denote the probability of rain.

1. Formulate this as a problem of decision making under uncertainty.
2. Can you identify an alternative that you would never choose, irrespective of the value of p_{rain}?
3. Can you write down of a rule that shows what the best thing to do is, as a function of p_{rain}? (The rule would say something like "if $p_{rain} \in [X, Y]$ then do Z, else if...")

Solution: Let F, P, and R abbreviate football, pub, and rowing, respectively, and let R and D abbreviate rain and dry, respectively. For notation convenience we write XY for the pair (X, Y).

(a) We specify the setting as a tuple $\langle \Omega, u \colon \Omega \to \mathbb{R}, \Sigma, g \colon \Sigma \to \mathit{lott}(\Omega) \rangle$, where p probability of rain.

- *Outcomes*:

$$\Omega = \{\text{football}, \text{pub}, \text{rowing}\} \times \{\text{rain}, \text{dry}\} = \{FR, FD, PR, PD, RR, RD\}$$

- The *utility function* u is given by:

$$\begin{aligned} u(FR) &= 1 & \qquad u(FD) &= 2 \\ u(PR) &= 3 & \qquad u(PD) &= 0 \\ u(RR) &= 0 & \qquad u(RD) &= 1 \end{aligned}$$

- *Strategies*: $\Sigma = \{\text{football}, \text{pub}, \text{rain}\} = \{F, P, R\}$
- The *outcome function* g is given by:

$$\begin{aligned} g(F) &= p * FR + (1 - p) * FD \\ g(P) &= p * PR + (1 - p) * PD \\ g(R) &= p * RR + (1 - p) * RD \end{aligned}$$

(b) Rowing, because football is preferred to rowing if it rains and football is preferred to rowing when it is dry. It then follows that football will be preferred in expectation as well. Formally, for every $p \in [0, 1]$

$$\begin{aligned} EU(g(R)) &= u(RR) * P(RR, g(R)) + u(RD) * P(RD, g(R)) \\ &= 0 * p + 1 * (1 - p) \\ &< 1 * p + 2 * (1 - p) \\ &= u(FR) * P(FR, g(F)) + u(FD) * P(FD, g(F)) \\ &= EU(g(F)) \end{aligned}$$

(c) By (b) you can abstract away from rowing. Then,

If $p \geq \frac{1}{2}$ we go pubbing, else play football.

Observe that we have:

$$EU(g(P)) \geq EU(g(F)) \quad \text{iff} \quad p \geq \tfrac{1}{2}.$$

To see this, consider the following equivalences:

$EU(g(P)) \geq EU(g(F))$
iff $u(PR) * (PR, g(P)) + u(PD) * P(PD, g(P)) \geq u(FR) * P(FR, g(F)) + u(FD) * P(FD, g(F)))$
iff $3p + 0(1 - p) \geq 1 * p + 2(1 - p)$
iff $3p \geq p + 2 - 2p$
iff $p \geq \frac{1}{2}$.

Exercise 3: Expected utility Suppose a person whose preferences satisfy the Von Neumann and Morgenstern axioms says that with respect to lotteries ℓ_1, ℓ_2, ℓ_3, and ℓ_4, her preferences are

$$\ell_1 \succeq \ell_2 \quad \text{and} \quad \ell_3 \succeq \ell_4.$$

Consider the following property. For all $p \in [0, 1]$ we have

$$p\ell_1 + (1 - p)\ell_3 \ \succeq \ p\ell_2 + (1 - p)\ell_4$$

Do you think this property should hold? If so, can you prove it using the von Neumann and Morgenstern axioms? '

Solution: The property should hold: As $\ell_1 \succeq \ell_2$, by *independence* also

$$p\ell_1 + (1 - p)\ell_3 \succeq p\ell_2 + (1 - p)\ell_3.$$

As $\ell_3 \succeq \ell_4$, by *independence* (and "mixing with $p\ell_2$") also

$$(1 - p)\ell_3 + p\ell_2 \succeq (1 - p)\ell_4 + p\ell_2.$$

Observe,

$$\begin{aligned} (1 - p)\ell_3 + p\ell_2 &= p\ell_2 + (1 - p)\ell_3 \\ (1 - p)\ell_4 + p\ell_2 &= p\ell_2 + (1 - p)\ell_4. \end{aligned}$$

By *transitivity* of $\succeq$, we finally obtain:

$$p\ell_1 + (1 - p)\ell_3 \succeq p\ell_2 + (1 - p)\ell_4$$

Exercise 4: Expected utility Recall the simple setting of win-lose lotteries. We have $\Omega = \{\mathcal{W}, \mathcal{L}\}$, and a preference relation $\succeq$ such that $\mathcal{W} \succ \mathcal{L}$. The continuity axiom for $\mathcal{W}$-$\mathcal{L}$ lotteries is as follows: $\ell_1 \succeq \ell_2$ iff $P(\mathcal{W}, \ell_1) \geq P(\mathcal{W}, \ell_2)$.

Show that if a preference relation over win-lose lotteries satisfies reflexity, totality, transitivity, and continuity, then there exists a von Neumann and Morgenstern utility function $u : \{\mathcal{W}, \mathcal{L}\} \to \mathbb{R}$ such that $EU(\ell_1) \geq EU(\ell_2)$ iff $\ell_1 \succeq \ell_2$, where the expected utility function $EU : \mathit{lott}(\{\mathcal{W}, \mathcal{L}\}) \to \mathbb{R}$ is defined as usual, with respect to the von Neumann and Morgenstern utility function u:

$$EU(\ell) = \sum_{\omega \in \{\mathcal{W}, \mathcal{L}\}} u(\omega) P(\omega, \ell)$$

Solution: Define $u \colon \{\mathcal{W}, \mathcal{L}\} \to \mathbb{R}$ such that

$$u(\omega) = \begin{cases} 1 & \text{if } \omega = \mathcal{W} \\ 0 & \text{if } \omega = \mathcal{L}. \end{cases}$$

Then,

$$\begin{aligned} EU(\ell_1) &= u(\mathcal{W}) P(\mathcal{W}, \ell_1) + u(\mathcal{L}) P(\mathcal{L}, \ell_1) \\ &= 1 P(\mathcal{W}, \ell_1) + 0 \\ &= P(\mathcal{W}, \ell_1) \end{aligned}$$

and

$$\begin{aligned} EU(\ell_2) &= u(\mathcal{W}) P(\mathcal{W}, \ell_2) + u(\mathcal{L}) P(\mathcal{L}, \ell_2) \\ &= 1(\mathcal{W}, \ell_2) + 0 \\ &= P(\mathcal{W}, \ell_2) \end{aligned}$$

Hence,

$$\ell_1 \succeq \ell_2 \Leftrightarrow_{\text{cont.}} P(\mathcal{W}, \ell_1) \geq P(\mathcal{W}, \ell_2) \Leftrightarrow_{\text{def. } EU} EU(\ell_1) \geq EU(\ell_2).$$

Exercise 5: Expected utility Consider the following scenario.

> You toss a fair coin repeatedly, until the coin shows heads for the first time. You are then paid $£2^k$, where k is the number of times the coin was tossed. For example, if the sequence of coin tosses was just H (i.e., heads appeared on the first toss) then $k = 1$ and you are paid $£2^1 = £2$. If the sequence was TTH then $k = 3$ and you are paid $£2^3 = £8$.

1. Express this as a lottery (in which the set of outcomes is infinite).
2. Assuming utility is expected monetary reward, compute the expected utility of this lottery.
3. Now assume that the utility function u over monetary outcomes is such that $u(£n) = \log_2 n$. Show that the agent's expected utility of this game is upward bounded.

4. (Discussion question, no marks given.) How much would *you* be willing to pay to enter this lottery?

5. (Discussion question, no marks given.) What do you conclude about the relationship between expected monetary reward and utility?

Solution:

1. Let $\Omega = \{H, TH, TTH, TTTH, \ldots\}$. Then,

$$\ell_{\text{St Petersburg}} = \tfrac{1}{2}H + \tfrac{1}{4}TH + \tfrac{1}{8}TTH + \cdots + \tfrac{1}{2^k}\underbrace{T\cdots T}_{k+1 \text{ times}}H + \ldots$$

2. If money=utility:

$$u(T^k H) = 2^{k+1}$$

Then,

$$\begin{aligned} EU(\ell_{\text{St Petersburg}}) &= u(H) * \tfrac{1}{2} + u(TH) * \tfrac{1}{4} + \ldots \\ &= 2^1 * \tfrac{1}{2} + 2^2 * \tfrac{1}{4} + \ldots \\ &= 1 + 1 + 1 + 1 + \ldots \\ &= \infty \end{aligned}$$

3. Instead of having a linear utility function, a (risk averse) logarithmic utility function over money could look like:

$$u'(\pounds k) = \log_2(k)$$

Then,

$$\begin{aligned} EU(\ell_{\text{St Petersburg}}) &= u'(H) * \tfrac{1}{2} + u'(TH) * \tfrac{1}{4} + \ldots \\ &= \log_2(2^1) * \tfrac{1}{2} + \log_2(2^2) * \tfrac{1}{4} + \ldots \\ &= 2 \end{aligned}$$

The expected utility of the lottery would thus be bounded by 2. This would correspond to a monetary value of $\pounds 4$.

Exercise 6: Expected utility The following story, albeit slightly morbid, is nevertheless apparently true.

> In the second world war, a US bomber squadron was based 3000km from their target. The target was so far away that fighter plane escorts were impossible, making missions even more than usually dangerous. Planes could only carry a few bombs on each mission, so that they could carry enough fuel to return to base. Pilots were scheduled to fly 30 missions before returning to the USA, but on average only half of the pilots survived all 30 missions.
>
> Logistics experts came up with the following proposal. Each plane would carry a much heavier bomb load – but only enough fuel to fly one way. Thus, each mission would be a suicide mission. However, the increased bomb load would mean that far fewer missions would be needed, allowing

75% of the pilots to return home. The other 25% of pilots, who had to fly the missions, would face certain death. Those to fly the missions would be selected randomly.

Every pilot who was presented with the new proposal rejected it in favour of the status quo.

1. Formulate the above two scenarios as lotteries within the von Neumann and Morgenstern framework, in which there are just two outcomes, $live$ and die, and such that $live \succ die$.

   Do the pilots satisfy von Neumann and Morgenstern's axioms? That is, does there exist a preference relation $\succeq$ over lotteries satisfying the von Neumann and Morgenstern axioms which is consistent with the pilot's preferences?

2. Now assume there are four outcomes, with associated preferences as follows:

   $$\text{live with honour} \succ \text{die with honour} \succ \text{live without honour}$$

   So: living with honour would mean flying a mission and surviving; to die with honour would mean flying a mission and being killed; and living without honour would mean living because somebody else had flown a mission to certain death.

   Reformulate the above scenarios as lotteries using these preferences. For this case, does there exist a preference relation $\succeq$ over lotteries satisfying the von Neumann and Morgenstern axioms which is consistent with the pilot's preferences?

3. (Discussion question, no marks given.) What factors do you think may have influenced the pilot's decisions?

Solution:

1. Let A be the status quo scenario and B the scenario in which the planes fly suicide missions. Then we can model the setting as a decision problem under uncertainty $\langle \Omega, u\colon \Omega \to \mathbb{R}, \Sigma, g\colon \Sigma \to lott(\Omega) \rangle$ where:

   - $\Omega = \{live, die\}$
   - $u(live) = 1 \qquad u(die) = 0$
   - $\Sigma = \{A, B\}$
   - $g(A) = \frac{1}{2} live + \frac{1}{2} die$
     $g(B) = \frac{3}{4} live + \frac{1}{4} die$

   Clearly, $EU(g(A)) < EU(g(B))$, even if utilities are defined very differently.

   Framed this way the pilots violated the **monotonicity** axiom: we have $live \succeq die$ and $\frac{3}{4} \geq \frac{1}{2}$, but $\frac{3}{4} live + \frac{1}{4} die \not\succeq \frac{1}{2} live + \frac{1}{2} die$.

   The pilots also violate either **independence** or **equivalence**. Assume for contradiction that their preferences satisfy both independence and equivalence. Then we could reason as follows: As $live \succeq die$, by independence,

   $$\frac{1}{2} live + \frac{1}{2} die \succeq \frac{1}{2} die + \frac{1}{2} die.$$

By another application of independence, we obtain

$$\tfrac{1}{2}(\tfrac{1}{2}\mathit{live} + \tfrac{1}{2}\mathit{die}) + \tfrac{1}{2}\mathit{live} \succeq \tfrac{1}{2}(\tfrac{1}{2}\mathit{die} + \tfrac{1}{2}\mathit{die}) + \tfrac{1}{2}\mathit{live}.$$

Finally, equivalence yields $\frac{3}{4}\mathit{live} + \frac{1}{4}\mathit{die} \succeq \frac{1}{2}\mathit{live} + \frac{1}{2}\mathit{die}$, *quod non.*

2. Also let $\Omega' = \{\textit{live-honour}, \textit{live-no-honour}, \textit{die-honour}\}$. Then, let A' and B' be the scenarios such that

$$g(A') = \tfrac{1}{2}\textit{live-honour} + \tfrac{1}{2}\textit{die-honour}$$
$$g(B') = \tfrac{3}{4}\textit{live-no-honour} + \tfrac{1}{4}\textit{die-honour}$$

The pilots do not necessarily violate the von Neumann Morgenstern axioms. To show this you have to produce a model, in this case a preference relation over all lotteries over Ω' that is consistent with $g(A') \succ g(B')$.

One solution is to define a utility function. Let a utility function $u\colon \Omega \to \mathbb{R}$ be given by

$$u(\textit{live-honour}) = 1$$
$$u(\textit{die-honour}) = \tfrac{1}{2}$$
$$u(\textit{live-no-honour}) = 0$$

and assume the pilots' preferences over *lotteries* over Ω be such that they prefer those lotteries ℓ that maximise the expected utility $EU(\ell)$ with respect to this utility function. It can easily be appreciated that this is consistent with the preferences over the outcomes Ω

$$\textit{live-honour} \succ \textit{die-honour} \succ \textit{live-no-honour}.$$

These preferences over lotteries over Ω' also satisfy all the von Neumann-Morgenstern axioms. Finally, we find that

$$\begin{aligned} EU(g(A')) &= \tfrac{1}{2}u(\textit{live-honour}) + \tfrac{1}{2}u(\textit{die-honour}) \\ &= \tfrac{3}{4} \\ &> \tfrac{1}{8} \\ &= \tfrac{3}{4}u(\textit{live-no-honour}) + \tfrac{1}{4}u(\textit{die-honour}) \\ &= EU(g(B')). \end{aligned}$$

Remark 1: This reasoning works for every choice of $u(\textit{die-honour}) \in [0, 1]$.

Another solution is to use the independence axiom and equivalence axioms. As *live-honour* $\succsim$ *die-honour*, by independence also

$$\tfrac{1}{2}\textit{live-honour} + \tfrac{1}{2}\textit{die-honour} \succeq \tfrac{1}{2}\textit{die-honour} + \tfrac{1}{2}\textit{die-honour} = \textit{die-honour}$$

Moreover, since *die-honour* $\succsim$, by another application of independence

$$\textit{die-honour} = \tfrac{3}{4}\textit{die-honour} + \tfrac{1}{4}\textit{die-honour} \succeq \tfrac{3}{4}\textit{live-no-honour} + \tfrac{1}{4}\textit{die-honour}.$$

The equivalence axiom and transitivity then give the result.

Remark 2: The second solution, however, does not yield a strict preference

$$\tfrac{1}{2}\textit{live-honour} + \tfrac{1}{2}\textit{die-honour} \succ \tfrac{3}{4}\textit{live-no-honour} + \tfrac{1}{4}\textit{die-honour}.$$

Remark 3: The pilots need not even violate the von Neumann-Morgenstern axioms if their preferences over Ω' were given by

$$\textit{live-honour} \succ \textit{live-no-honour} \succ \textit{die-honour}.$$

Why?

Exercise 7: von Neumann-Morgenstern utility functions Let $\succeq$ be a rational, continuous, and independent preference relation on $lott(\{\omega_1, \ldots, \omega_k\})$ and $u\colon \{\omega_1, \ldots, \omega_k\} \to \mathbb{R}$ a utility function representing $\succeq$, i.e., for any two simple lotteries $\ell_1 = p_1\omega_1 + \cdots + p_k\omega_k$ and $\ell_2 = q_1\omega_1 + \cdots + q_k\omega_k$,

$$\ell_1 \succeq \ell_2 \text{ if and only if } \sum_{i=1}^{k} u(\omega_i)P(\omega_i, \ell_1) \geq \sum_{i=1}^{k} u(\omega_i)P(\omega_i, \ell_2).$$

1. Show that for $a, b \in \mathbb{R}$, $a > 0$, the utility function $u'\colon \omega_i \mapsto au(\omega_i) + b$ also represents $\succeq$.
2. Is it the case that for *all* strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$, the utility function $u'' : \omega_i \mapsto f(u(\omega_1))$ also represents $\succeq$?

Solution:

1. By rewriting and reorganising terms. As von Neumann-Morgenstern utility functions satisfy equivalence, it suffices to prove that for any two lotteries $\ell_1 = p_1\omega_1 + \cdots + p_k\omega_k$ and $\ell_2 = q_1\omega_1 + \cdots + q_k\omega_k$,

$$\ell_1 \succeq \ell_2 \text{ if and only if } \sum_{i=1}^{k} u'(\omega_i)P(\omega_i, \ell_1) \geq \sum_{i=1}^{k} u'(\omega_i)P(\omega_i, \ell_2).$$

Consider the following equivalences:

$$\begin{aligned}
&\ell_1 \succeq \ell_2 \\
\text{iff}\quad & \sum_{i=1}^{k} u(\omega_i)P(\omega_i, \ell_1) \geq \sum_{i=1}^{k} u(\omega_i)P(\omega_i, \ell_2) \\
\text{iff}\quad & a\Big(\sum_{i=1}^{k} u(\omega_i)P(\omega_i, \ell_1)\Big) + b \geq a\Big(\sum_{i=1}^{k} u(\omega_i)P(\omega_i, \ell_2)\Big) + b \\
\text{iff}\quad & \sum_{i=1}^{k} au(\omega_i)P(\omega_i, \ell_1) + \sum_{\omega\in\Omega} bP(\omega_i, \ell_1) \geq \sum_{i=1}^{k} au(\omega_i)P(\omega_i, \ell_2) + \sum_{\omega\in\Omega} bP(\omega_i, \ell_2) \\
\text{iff}\quad & \sum_{i=1}^{k} \Big(au(\omega_i)P(\omega_i, \ell_1) + bP(\omega_i, \ell_1)\Big) \geq \sum_{i=1}^{k} \Big(au(\omega_i)P(\omega_i, \ell_2) + bP(\omega_i, \ell_2)\Big) \\
\text{iff}\quad & \sum_{i=1}^{k} (au(\omega_i) + b)P(\omega_i, \ell_1) \geq \sum_{i=1}^{k} (au(\omega_i) + b)P(\omega_i, \ell_2) \\
\text{iff}\quad & \sum_{i=1}^{k} u'(\omega_i)P(\omega_i, \ell_1) \geq \sum_{i=1}^{k} u'(\omega_i)P(\omega_i, \ell_2)
\end{aligned}$$

2. Consider lotteries ℓ_1, ℓ_2, ℓ_3 and $\frac{1}{2}\ell_1 + \frac{1}{2}\ell_3$. Assume that

$$u(\ell_1) = 0 \qquad u(\ell_2) = 1 \qquad u(\ell_3) = 2,$$

Also assume for contradiction, that both u and u' are von Neumann-Morgenstern utility functions representing $\succeq$. Then,

$$EU(\tfrac{1}{2}\ell_1 + \tfrac{1}{2}\ell_3) = EU(\ell_2)$$

and hence,

$$\tfrac{1}{2}\ell_1 + \tfrac{1}{2}\ell_3 \sim \ell_2. \qquad (*)$$

Moreover,

$$u'(\ell_2) = 1^2 = 1,$$

whereas

$$EU'(\ell_1 + \ell_3) = (u(\ell_1)^2 \cdot \tfrac{1}{2}) + (u(\ell_3)^2 \cdot \tfrac{1}{2}) = 0 + 2 = 2,$$

where $EU'(\ell)$ denotes $\sum_{\omega \in \Omega} u'(\omega)P(\omega, \ell)$. It would follow that

$$\tfrac{1}{2}\ell_1 + \tfrac{1}{2}\ell_3 \succ \ell_2,$$

which, however, contradicts $(*)$.