Tutorial 2: Strategic Form Games and Mixed Strategies (Solutions)

Exercise 1: Formulating a decision scenario as a strategic game: This question relates to a (genuine!) British TV show called called "GoldenBalls".

(a) Watch the following video:

`http://tinyurl.com/pds7xxn`

Formulate this as a payoff matrix, assuming utility is monetary reward, and analyse it using the solution concepts and social welfare concepts introduced in the lecture.

(b) (For discussion, no marks awarded.) Apart from monetary reward, what other factors might play a role in the formulation of utility here? Can you refine your game to reflect these?

(c) (For discussion, no marks awarded.) Now watch the following video:

`http://tinyurl.com/ofruebv`

What do you think is going on here?

Solution:

(a) The situation can be modelled by the following strategic game:

	split	*steal*
split	£50075, £50075	£0, £100150
steal	£100150, £0	£0, £0

- dominant strategy: for both players *steal* is a dominant strategy (*split* is not).
- dominant strategy equilibrium: both *steal*
- Nash equilibria: all but (*split*,*split*)
- Social welfare
  - utilitarian social welfare: all but $(steal, steal)$
  - Pareto optimal outcomes: all but $(steal, steal)$
  - egalitarian social welfare: $(split, split)$

(b) Other factors than monetary gain that can play a role when playing this game:

- regret, not wanting to be a "sucker", gullible, or exploited.
- spite
- loyalty, honesty, feeling of fairness
- past experiences
- reputation may play a role as millions of people watching
- belief what the other will do (for other reasons than expected win) and attitude towards risk
- the setting had perhaps better be modelled as a more cooperative setting. (The makers of the show have taken pains to make the game as non-cooperative as possible, but perhaps the preplay communication and the social pressure make that in a sense binding agreements (at the punishment of social derision) can be made.)

(c) Points to consider:

- **Commitment**: Nick is trying to commit himself playing *steal*, thereby trying to improve his strategic position.
- Nick is trying to make the situation asymmetric
- Reasons why Nick may have decided to play *split* after having said being committed to *steal*:
  - Thus, he puts moral pressure on Ibraham to split after the show in case he would have (irrationally) played *steal* after all.
  - The commitment to split after the show, is thus implemented in a more direct and public way.

Exercise 2: Solution concepts and social welfare concepts Either prove or disprove each of the following statements in the context of 2×2 games (disproving is usually best done with a counter-example):

(a) *If a player i has a dominant strategy in a game, then in every Nash equilibrium of that game player i will choose a dominant strategy.*

Solution: Does not hold. In the following game (B, R) is a Nash equilibrium although B and R are no dominant strategies. Moreover, T and L are (weakly) dominant.

	L	R
T	$\underline{1, 1}$	$0, 0$
B	$0, 0$	$\underline{0, 0}$

Observe that the claim does hold, if a player has a strategy that *strictly* dominates all others, as she can always deviate to this strictly dominant action.

(b) *If a game has a dominant strategy equilibrium, then it is unique: the game has no other dominant strategy equilibria.*

Solution: Does not hold. In the following game every strategy is (weakly) dominant and every strategy profile a dominant strategy equilibrium.

	L	R
T	$\underline{0,0}$	$\underline{0,0}$
B	$\underline{0,0}$	$\underline{0,0}$

A slightly more interesting solution would be the following:

	L	R
T	$\underline{1,1}$	$0,0$
B	$0,0$	$\underline{0,0}$

(B, R) is also a Nash equilibrium even though (T, L) is the only dominant strategy equilibrium. Again, observe an outcome in which all players play a strictly dominant strategy, i.e., a strategy that strictly dominates all other strategies of that player, will be unique.

(c) *Every dominant strategy equilibrium of a game is a Nash equilibrium.*

Solution: Holds. By contraposition. Assume that it is *not* a Nash equilibrium. Then, there is a player i and a strategy $\sigma_i' \in \Sigma_i$ such that

$$u_i(\vec{\sigma}_{-i}, \sigma_i') > u_i(\vec{\sigma}).$$

Hence, σ_i is *not* a dominant strategy and *a fortiori* $\vec{\sigma} = (\sigma_1, \sigma_2)$ is *not* a dominant strategy equilibrium. QED

Remark: Can also be proven directly, but is slightly less intuitive.

(d) *Every Nash equilibrium of a game is a dominant strategy equilibrium.*

Solution: Does not hold. See game in Exercise $2(a)$.

(e) *If a game outcome ω maximises utilitarian social welfare, then ω is Pareto efficient.*

Solution: Holds. Assume that ω maximises utilitarian social welfare and, for contradiction that ω is *not* Pareto efficient. Then there is an outcome ω' such that $u_i(\omega') \geq u_i(\omega)$ for all agents i and $u_j(\omega') > u_j(\omega)$ for some agent j. It follows that

$$u_1(\omega') + \cdots + u_n(\omega') > u_1(\omega) + \cdots + u_n(\omega),$$

contradicting that ω maximises utilitarian social welfare. QED

(f) *If a game outcome ω is Pareto efficient, then it maximises utilitarian social welfare.*

Solution: Does not hold. Consider the following game

	L	R
T	$1, 2$	$1, 1$
B	$1, 1$	$11, 1$

Here the outcome with payoffs $(1, 2)$ is Pareto efficient but obviously does not maximise utilitarian social welfare.

(g) *If all utilities in a game are positive, then any outcome that maximises the product of utilities of players is Pareto efficient.*

Solution: Holds. By the same argument as Exercise $2(e)$. Assume that ω maximises the product of agents' utilities and, for contradiction that ω is *not* Pareto efficient. Then there is an outcome ω' such that $u_i(\omega') \geq u_i(\omega)$ for all agents i and $u_j(\omega') > u_j(\omega)$ for some agent j. It follows that

$$u_1(\omega') \times \cdots \times u_n(\omega') > u_1(\omega) \times \cdots \times u_n(\omega),$$

contradicting that ω maximises the product of agents' utilities. QED

(h) *If all utilities in a game are positive, then any Pareto efficient outcome of the game will maximise the product of utilities of players.*

Solution: Does not hold. Again the game of Exercise $2(f)$.

Exercise 3: Nash equilibria in Mixed Strategies If we use mixed strategies in a game, then we are in the domain of expected utility.

1. Write down an expression for the expected utility of each player in a generic 2×2 game, when a mixed strategy is given as a pair $(p, q) \in [0, 1]^2$. That is, define the expressions $EU_1(p, q)$ and $EU_2(p, q)$.

2. Generalise the expression you obtained in the first part to n player games, where each player $i \in N$ has pure strategy set Σ_i. Denote a mixed strategy profile by $\vec{ms} = (ms_1, \ldots, ms_n)$, where $ms_i \in \Delta\Sigma_i$ is a mixed strategy for i, i.e., a

probability distribution over Σ_i. Use $ms_i(\sigma)$ to denote the probability of $\sigma \in \Sigma_i$ being played in the mixed strategy ms_i.

Hint: First define an expression $P(\vec{\sigma}, \vec{ms})$, meaning the probability that the pure strategies $\vec{\sigma}$ are chosen given the mixed strategy profile $\vec{ms}$. Then define expected utility using this expression.

Solution: Let the generic 2×2 game be given by

	L	R
T	ω_1	ω_2
B	ω_3	ω_4

We identify strategy profiles and outcomes and write, for instance, TL for ω_1. Moreover, p is the probability with which agent 1 plays strategy T and q the probability with which agent 2 plays strategy L. Playing a mixed strategy leads to an uncertain outcome or lottery ℓ_{pq} over the set Ω of outcomes ω_1, ω_2, ω_3, and ω_4. Then,

(a) We identify strategy profiles and outcomes and write, for instance, TL for ω_1. Moreover, p is the probability with which agent 1 plays strategy T and q the probability with which agent 2 plays strategy L. Playing a mixed strategy leads to an uncertain outcome or lottery ℓ_{pq} over the set Ω of outcomes ω_1, ω_2, ω_3, and ω_4:

$$\ell_{pq} = pq\omega_1 + p(1-q)\omega_2 + (1-p)q\omega_3 + (1-p)(1-q)\omega_4.$$

Then,

$$\begin{aligned} EU_1(p,q) &= EU_1(\ell_{pq}) \\ &= \sum_{\omega \in \Omega} u_1(\omega) \cdot P(\omega, \ell_{pq}) \\ &= pq \cdot u_1(\omega_1) + p(1-q) \cdot u_1(\omega_2) + (1-p)q \cdot u_1(\omega_3) + (1-p)(1-q) \cdot u_1(\omega_4) \\ EU_2(p,q) &= EU_2(\ell_{pq}) \\ &= \sum_{\omega \in \Omega} u_2(\omega) \cdot P(\omega, \ell_{pq}) \\ &= pq \cdot u_2(\omega_1) + p(1-q) \cdot u_2(\omega_2) + (1-p)q \cdot u_2(\omega_3) + (1-p)(1-q) \cdot u_2(\omega_4) \end{aligned}$$

(b) First define

$$P(\vec{\sigma}, \vec{ms}) = \prod_{i \in N} ms_i(\sigma_i)$$

Then the generalised expression for expected utility can be given by

$$EU_i(\vec{ms}) = \sum_{\vec{\sigma} \in \vec{\Sigma}} P(\vec{\sigma}, \vec{ms}) u_i(\vec{\sigma})$$

Writing things out fully, the generalised expression would look like this (the outer pair of parenthesis is only there for clarity, but can safely be omitted).

$$EU_i(\vec{ms}) = \sum_{\vec{\sigma} \in \vec{\Sigma}} \left(\left(\prod_{i \in N} ms_i(\sigma_i) \right) u_i(\vec{\sigma}) \right).$$

Exercise 4: Nash equilibria For each of the following games:

(i) identify any dominant strategies, dominant strategy equilibria, and pure Nash equilibria;

(ii) identify outcomes that are Pareto efficient, that maximise utilitarian social welfare, and that maximise egalitarian social welfare

(iii) apply the principle of indifference to identify any fully mixed Nash equilibria

(iv) compute the expected utility of each player for each fully mixed strategy equilibrium you identify.

Solution: For an overview of results also see the table in Figure 1. For part (iii) solve each time for player 1 the following scheme:

$$\begin{aligned} & EU_2(L,p) = EU_2(R,p) \\ \text{iff} \quad & pu_2(TL) + (1-p)u_2(BL) = pu_2(TR) + (1-p)u_2(BR) \\ \text{iff} \quad & p = \end{aligned}$$

and for player 2:

$$\begin{aligned} & EU_1(T,q) = EU_1(B,q) \\ \text{iff} \quad & qu_1(TL) + (1-q)u_2(TR) = qu_2(BL) + (1-q)u_1(BR) \\ \text{iff} \quad & q = . \end{aligned}$$

(a) Battle of the Sexes

	L	R
T	$2,1$	$0,0$
B	$0,0$	$1,2$

(i)
- dominant strategies: none
- dominant strategy equilibria: none
- pure Nash equilibria: (T,L) and (B,R)

(ii)
- Pareto efficient: (T,L) and (B,R)
- utilitarian social welfare maximizing: (T,L) and (B,R)
- egalitarian social welfare maximizing: (T,L) and (B,R) (if no mixed strategies)

(iii) For player 1:

$$\begin{aligned} & EU_2(L,p) = EU_2(R,p) \\ \text{iff} \quad & pu_2(TL) + (1-p)u_2(BL) = pu_2(TR) + (1-p)u_2(BR) \\ \text{iff} \quad & 1p + 0(1-p) = 0p + 2(1-p) \\ \text{iff} \quad & p = 2 - 2p \\ \text{iff} \quad & 3p = 2 \\ \text{iff} \quad & p = \tfrac{2}{3} \end{aligned}$$

For player 2:

$$\begin{aligned}
& EU_1(T,q) = EU_1(B,q) \\
\text{iff} \quad & qu_1(TL) + (1-q)u_2(TR) = qu_2(BL) + (1-q)u_1(BR) \\
\text{iff} \quad & 2q + 0(1-q) = 0q + 1(1-q) \\
\text{iff} \quad & 2q = 1 - q \\
\text{iff} \quad & 3q = 1 \\
\text{iff} \quad & q = \tfrac{1}{3}.
\end{aligned}$$

Hence, $(\frac{2}{3}, \frac{1}{3})$ is a fully mixed Nash equilibrium.

(iv) Expected utilities:

$$\begin{aligned}
EU_1(p,q) &= \tfrac{1}{3} \cdot 2 + \tfrac{2}{2} \cdot 0 = \tfrac{2}{3} \\
EU_2(p,q) &= \tfrac{2}{3} \cdot 0 + \tfrac{1}{3} \cdot 2 = \tfrac{2}{3}
\end{aligned}$$

(b) Stag Hunt

	L	R
T	$9,9$	$0,8$
B	$8,0$	$7,7$

(i)
- dominant strategies: none
- dominant strategy equilibria: none
- pure Nash equilibria: (T,L) and (B,R)

(ii)
- Pareto efficient: (T,L)
- utilitarian social welfare maximizing: (T,L)
- egalitarian social welfare maximizing: (T,L)

(iii) For player 1:

$$\begin{aligned}
& EU_2(L,p) = EU_2(R,p) \\
\text{iff} \quad & 9p + 0(1-p) = 8p + 7(1-p) \\
\text{iff} \quad & 9p = 8p + 7 - 7p \\
\text{iff} \quad & 8p = 7 \\
\text{iff} \quad & p = \tfrac{7}{8}
\end{aligned}$$

For player 2:

$$\begin{aligned}
& EU_1(T,q) = EU_1(B,q) \\
\text{iff} \quad & 9q + 0(1-q) = 8q + 7(1-q) \\
\text{iff} \quad & 9q = 8q + 7 - 7q \\
\text{iff} \quad & 9q = 8q + 7 - 7q \\
\text{iff} \quad & 8q = 7 \\
\text{iff} \quad & q = \tfrac{7}{8}
\end{aligned}$$

Hence, $(\frac{7}{8}, \frac{7}{8})$ is a fully mixed Nash equilibrium.

(iv) Expected utilities:

$$EU_1(p,q) = \frac{7}{8} \cdot 9 + \frac{1}{8} \cdot 0 = \frac{63}{8} = 7.875$$
$$EU_2(p,q) = \frac{7}{8} \cdot 9 + \frac{1}{8} \cdot 0 = \frac{63}{8} = 7.875$$

(c) This is a "Hawk-Dove" game

	L	R
T	$4, 4$	$2, 8$
B	$8, 2$	$1, 1$

(i)
- dominant strategies: none
- dominant strategy equilibria: none
- pure Nash equilibria: (T, R) and (B, L)

(ii)
- Pareto efficient: (T, L) (B, L) and (T, R)
- utilitarian social welfare maximizing: (T, R) and (B, L)
- egalitarian social welfare maximizing: (T, L)

(iii) For player 1:

$$\begin{aligned}
& EU_2(L, p) = EU_2(R, p) \\
\text{iff} \quad & pu_2(TL) + (1-p)u_2(BL) = pu_2(TR) + (1-p)u_2(BR) \\
\text{iff} \quad & 4p + 2(1-p) = 8p + 1(1-p) \\
\text{iff} \quad & 4p + 2 - 2p = 8p + 1 - p \\
\text{iff} \quad & 2p + 2 = 7p + 1 \\
\text{iff} \quad & 5p = 1 \\
\text{iff} \quad & p = \tfrac{1}{5}
\end{aligned}$$

For player 2:

$$\begin{aligned}
& EU_1(T, q) = EU_1(B, q) \\
\text{iff} \quad & qu_1(TL) + (1-q)u_2(TR) = qu_2(BL) + (1-q)u_1(BR) \\
\text{iff} \quad & 4q + 2(1-q) = 8q + 1(1-q) \\
\text{iff} \quad & 4q + 2 - 2q = 8q + 1 - q \\
\text{iff} \quad & 2q + 2 = 7q + 1 \\
\text{iff} \quad & 5q+ = 1 \\
\text{iff} \quad & q = \tfrac{1}{5}.
\end{aligned}$$

Hence, $(\frac{1}{5}, \frac{1}{5})$ is a fully mixed Nash equilibrium.

(iv) Expected utility

$$EU_1(p,q) = \frac{1}{5} \cdot 4 + \frac{4}{5} \cdot 2 = \frac{12}{5} = 2\frac{2}{5}$$
$$EU_2(p,q) = \frac{1}{5} \cdot 4 + \frac{4}{5} \cdot 2 = \frac{12}{5} = 2\frac{2}{5}$$

(d)

	L	R
T	$0,0$	$3,5$
B	$4,4$	$0,3$

(i)
- dominant strategies: none
- dominant strategy equilibria: none
- pure Nash equilibria: (T,R) and (B,L)

(ii)
- Pareto efficient: (T,R) and (B,L)
- utilitarian social welfare maximizing: (T,R) and (B,L)
- egalitarian social welfare maximizing: (B,L)

(iii) For player 1:

$$\begin{aligned}
& EU_2(L,p) = EU_2(R,p) \\
\text{iff} \quad & pu_2(TL) + (1-p)u_2(BL) = pu_2(TR) + (1-p)u_2(BR) \\
\text{iff} \quad & 0p + 4(1-p) = 5p + 3(1-p) \\
\text{iff} \quad & 4 - 4p = 5p + 3 - 3p \\
\text{iff} \quad & 6p = 1 \\
\text{iff} \quad & p = \tfrac{1}{6}
\end{aligned}$$

For player 2:

$$\begin{aligned}
& EU_1(T,q) = EU_1(B,q) \\
\text{iff} \quad & qu_1(TL) + (1-q)u_2(TR) = qu_2(BL) + (1-q)u_1(BR) \\
\text{iff} \quad & 0q + 3(1-q) = 4q + 0(1-q) \\
\text{iff} \quad & 3 - 3q = 4q \\
\text{iff} \quad & 7q = 3 \\
\text{iff} \quad & q = \tfrac{3}{7}.
\end{aligned}$$

Hence, $(\frac{1}{6}, \frac{3}{7})$ is a fully mixed Nash equilibrium.

(iv) Expected utility

$$\begin{aligned}
EU_1(p,q) &= \tfrac{3}{7} \cdot 0 + \tfrac{4}{7} \cdot 3 = \tfrac{12}{7} = 1\tfrac{5}{7} \\
EU_2(p,q) &= \tfrac{1}{6} \cdot 0 + \tfrac{5}{6} \cdot 4 = \tfrac{20}{6} = \tfrac{10}{3} = 3\tfrac{1}{3}
\end{aligned}$$

(e)

	L	R
T	$1,1$	$4,0$
B	$2,10$	$3,5$

(i)
- dominant strategies: for column player L
- dominant strategy equilibria: none
- pure Nash equilibria: (B, L)

(ii)
- Pareto efficient: (T, R), (B, L) and (B, R) (all but (T, L))
- utilitarian social welfare maximizing: (B, L)
- egalitarian social welfare maximizing: (B, R)

(iii) For player 1:

$$
\begin{aligned}
& EU_2(L, p) = EU_2(R, p) \\
\text{iff} \quad & pu_2(TL) + (1-p)u_2(BL) = pu_2(TR) + (1-p)u_2(BR) \\
\text{iff} \quad & 1p + 10(1-p) = 0p + 5(1-p) \\
\text{iff} \quad & p + 10 - 10p = 5 - 5p \\
\text{iff} \quad & 4p = 5 \\
\text{iff} \quad & p = 1\tfrac{1}{4}
\end{aligned}
$$

In this case the equations do not have a solution in $[0, 1]$, and, therefore, p is no probability distribution. We may conclude that there is *no* fully mixed equilibrium.

(iv) Not applicable. There is no fully mixed Nash equilibrium.

	dom.strs.	dom eq.	PNE	PO	Util.	Egal.	fully MNE	EU
a	$-$	$-$	$(T,L),(B,R)$	$(T,L),(B,R)$	$(T,L),(B,R)$	$(T,L),(B,R)$	$(\frac{2}{3},\frac{1}{3})$	$(\frac{2}{3},\frac{2}{3})$
b	$-$	$-$	$(T,L),(B,R)$	(T,L)	(T,L)	(T,L)	$(\frac{7}{8},\frac{7}{8})$	$(\frac{63}{8},\frac{63}{8})$
c	$-$	$-$	$(T,R),(B,L)$	$(T,L),(T,R),(B,L)$	$(T,R),(B,L)$	(T,L)	$(\frac{1}{5},\frac{1}{5})$	$(\frac{12}{5},\frac{12}{5})$
d	$-$	$-$	$(T,R),(B,L)$	$(T,R),(B,L)$	$(T,R),(B,L)$	(B,L)	$(\frac{1}{6},\frac{3}{7})$	$(\frac{12}{7},\frac{10}{3})$
e	L	$-$	(B,L)	$(T,R),(B,L),(B,R)$	(B,L)	(B,R)	$-$	$-$

Figure 1: Table of results for Exercise 4

Exercise 5: Mixed Nash Equilibria and the Indifference Principle This question refers to the "generic" 2×2 game that we discussed in the lecture, where the row players has pure strategies T and B, and the column player has strategies L and R. Prove that a pair of probabilities $(p, q) \in (0, 1)^2$ is a mixed strategy Nash equilibrium in the generic 2×2 game iff:

$$\begin{aligned} EU_1(T, q) &= EU_1(B, q) \quad \text{and} \\ EU_2(L, p) &= EU_2(R, p) \end{aligned}$$

Solution: Intuitively, $EU_1(T, q)$ is the utility player 1 can expect when playing T against the mixed strategy q of player 2. Now observe that:

$$\begin{aligned} EU_1(p,q) &= pqu_1(T,L) + p(1-q)u_1(T,R) + (1-p)qu-1(B,L) + (1-p)(1-q)u_1(B,R) \\ &= p(qu_1(T,L) + (1-q)u_1(T,R)) + (1-p)(qu-1(B,L) + (1-q)u_1(B,R)) \\ &= p(EU_1(T,q)) + (1-p)(EU_1(B,q)) \\ EU_2(p,q) &= q(EU_1(L,p)) + (1-q)(EU_2(R,p)) \end{aligned}$$

First first assume that

$$EU_1(T, q) = EU_1(B, q),$$

and we show that player 1 does not want to deviate to any other pure or mixed strategy. To this end consider an arbitrary $p' \in [0, 1]$. We show that:

$$EU_1(p, q) \geq EU_1(p', q).$$

Consider the following identities:

$$\begin{aligned} EU_1(p,q) &= p(EU_1(T,q)) + (1-p)(EU_1(B,q)) \\ &= p(EU_1(T,q)) + (1-p)(EU_1(T,q)) \\ &= EU_1(T,q) \\ &= p'(EU_1(T,q)) + (1-p')(EU_1(T,q)) \\ &= p'(EU_1(T,q)) + (1-p')(EU_1(B,q)) \\ &= EU_1(p',q). \end{aligned}$$

It follows that player 1 does not want to deviate from mixed strategy p. The argument for the second player is analogous and we may conclude that $p \in (0, 1)$ is a fully mixed Nash equilibrium.

For the opposite direction, assume that $(p, q) \in (0, 1)^2$ is a fully mixed Nash equilibrium and, for contradiction, that

$$EU_1(T, q) \neq EU_1(B, q) \quad \text{or} \quad EU_2(L, p) \neq EU_2(R, p).$$

Without loss of generality, we may assume the former and that

$$EU_1(T, q) > EU_1(B, q).$$

Now consider the following inequalities, which shows that in this case player 1 had better play T with probability 1:

$$\begin{aligned} EU_1(p,q) &= pEU_1(T,q) + (1-p)EU_1(B,q) \\ &< pEU_1(T,q) + (1-p)EU_1(T,q) \\ &= EU_1(T,q). \end{aligned}$$

(Here, T also denotes the mixed strategy in which T is played with pobability 1.) It follows that (p, q) is *not* a Nash equilibrium.

Exercise 6: Minimax theorem in zero sum games

(For greater glory, no marks given.)

Prove the minimax theorem (for pure strategies only), as stated in the lecture.

Solution: Recall the definitions of the maximin and minimax values, which are as follows.

$$\overline{v} = \max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2)$$
$$\underline{v} = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2)$$

We first prove that $\overline{v} \leq \underline{v}$. This does not holds for zero-sum games only but for strategic games in general!

Proof: Consider $\overline{v}$ and let (τ_1, τ_2) "instantiate" $\overline{v}$, that is,

$$\tau_1 \in \operatorname*{argmax}_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2)$$

and

$$\tau_2 \in \operatorname*{argmin}_{\sigma_2 \in \Sigma_2} u_1(\tau_1, \sigma_2).$$

Then, $u_1(\tau_1, \tau_2) = \overline{v}$. Moreover,

$$\text{for all } \sigma_2' \in \Sigma_2: \quad u_1(\tau_1, \tau_2) \leq u_1(\tau_1, \sigma_2').$$

Therefore also,

$$\text{for all } \sigma_2' \in \Sigma_2: \quad u_1(\tau_1, \tau_2) \leq \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2').$$

This in particular holds for the $\sigma_2' \in \Sigma_2$ that minimises $\max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2')$. Hence,

$$\overline{v} = u_1(\tau_1, \tau_2) \leq \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2) = \underline{v}.$$

This concludes the proof. □

Now we are in a position to prove the minimax theorem as in the slides.

Theorem. *Suppose we have a two-player zero-sum game, in which (σ_1, σ_2) is a Nash equilibrium. Then,*

$$u_1(\sigma_1, \sigma_2) = \overline{v} = \underline{v}$$

Thus, in two-player zero-sum games, Nash equilibria and maximin-minimax outcomes coincide.

Proof: Let $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a *pure* Nash equilibrium in a two-player zero-sum game. As generally $\overline{v} \leq \underline{v}$, it suffices to show that

$$u_i(\sigma_1^*, \sigma_2^*) \leq \overline{v} \text{ and } \underline{v} \leq u_i(\sigma_1^*, \sigma_2^*).$$

First assume for contradiction that

$$u_1(\sigma_1^*, \sigma_2^*) > \overline{v}.$$

Let $\sigma_2' \in \operatorname{argmin}_{\sigma_2 \in \Sigma_2} u_1(\sigma_1^*, \sigma_2)$. Then,

$$u_1(\sigma_1^*, \sigma_2') \leq \overline{v} < u_1(\sigma_1^*, \sigma_2^*).$$

As the game is zero-sum, then,

$$u_2(\sigma_1^*, \sigma_2') \geq \overline{v} > u_2(\sigma_1^*, \sigma_2^*).$$

It follows that σ_2^* is not a best response to σ_1^* and that (σ_1^*, σ_2^*) is not a pure Nash equilibrium, a contradiction.

Now assume for contradiction that

$$u_1(\sigma_1^*, \sigma_2^*) < \underline{v}.$$

Let $\sigma_1' \in \operatorname{argmax}_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2^*)$. Then,

$$u_1(\sigma_1', \sigma_2^*) \geq \underline{v} > u_1(\sigma_1^*, \sigma_2^*).$$

It follows that σ_1^* is not a best response to σ_2^* and that (σ_1^*, σ_2^*) is not a pure Nash equilibrium, a contradiction. □